Answer Key

NAME:

Math 150 Exam 1

Instructions: WRITE YOUR NAME CLEARLY. Do as many problems as you can for a maximal score of 100. SHOW YOUR WORK!

1. True or False?

a) For any function f, $\lim_{x \to a} f(x) = f(a)$ [2 pts]

F

b) If $f(x) = 3^x$, then $f'(x) = x3^{x-1}$ [2 pts]

F

c) $\lim_{x \to 0} \frac{Sin(3x)}{x} = 3$ [2 pts]

T

d) $\frac{x^2 + x - 6}{x - 2} = x + 3$ for all $x \in (-\infty, \infty)$ [2 pts]

F

e) $\lim_{x \to 4} \frac{5 + \sqrt{x}}{\sqrt{5 + x}} = \frac{7}{3}$ [2 pts]

T

2. Let $f(x) = -x^2 + 3x - 2$. Write the equation of the tangent line to the graph of f(x) at x = 1. (Hint: Use derivative "shortcuts") [10 pts]

$f'(x) = -2x + 3$

$f'(1) = -2 + 3 = 1$

$y - y_1 = f'(1)(x-1)$ Hence

$y - f(1) = f'(1)(x-1)$ becomes

$y - 0 = 1(x-1)$ or $\boxed{y = x - 1}$

3. Evaluate $\lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right)$ [10 pts]

$$\lim_{x\to-\infty} x+\sqrt{x^2+2x} = \lim_{x\to-\infty} \frac{\left(x+\sqrt{x^2+2x}\right)\left(x-\sqrt{x^2+2x}\right)}{x-\sqrt{x^2+2x}}$$

$$= \lim_{x\to-\infty} \frac{x^2-(x^2+2x)}{x-\sqrt{x^2+2x}} = \lim_{x\to-\infty} \frac{-2x}{x-\sqrt{x^2+2x}} =$$

$$= \lim_{x\to-\infty} \frac{-2x\left(\frac{1}{x}\right)}{\left(x-\sqrt{x^2+2x}\right)\left(\frac{1}{x}\right)} = \lim_{x\to-\infty} \frac{-2}{1-\frac{\sqrt{x^2+2x}}{x}} =$$

$$= \lim_{x\to-\infty} \frac{-2}{1+\sqrt{\frac{x^2+2x}{x^2}}} = \lim_{x\to-\infty} \frac{-2}{1+\sqrt{1+\frac{2}{x}}} = \frac{-2}{1+\sqrt{1}} = -1$$

4. Let $f(x)=\sqrt{1-3x}$. Use the definition of the derivative to find $f'(x)$ [10 pts]

$$F'(x) = \lim_{h\to0} \frac{\sqrt{1-3(x+h)}-\sqrt{1-3x}}{h} =$$

$$= \lim_{h\to0} \frac{1-3(x+h)-(1-3x)}{h\left(\sqrt{1-3(x+h)}+\sqrt{1-3x}\right.} = \lim_{h\to0} \frac{-3h}{h\left(\sqrt{1-3(x+h)}+\sqrt{1-3x}\right)}$$

$$= \lim_{h\to0} \frac{-3}{\sqrt{1-3(x+h)}+\sqrt{1-3x}} = \frac{-3}{\sqrt{1-3x}+\sqrt{1-3x}} = \frac{-3}{2\sqrt{1-3x}}$$

5. Use the Quotient Rule to differentiate $K(x) = \dfrac{Cos(x)}{1 - Sin(x)}$ [10 pts]

$$K'(x) = \frac{(1-\sin x)(-\sin x) - (-\cos x)(\cos x)}{(1-\sin x)^2} =$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} = \frac{1-\sin x}{(1-\sin x)^2} =$$

$$= \frac{1}{1-\sin x}.$$

6. Suppose that $f(3) = -4$, $f'(3) = 1$, $g(3) = 5$, and $g'(3) = 2$. Compute $(fg)'(3)$. [10 pts]

$$(fg)'(3) = f'(3)g(3) + f(3)g'(3) =$$

$$= 1 \cdot 5 + (-4) \cdot 2 = 5 - 8 = -3$$

7. Evaluate $\lim_{x\to 0}\frac{Sin(2x)Sin(5x)}{x^2}$ [10 pts]

$$\lim_{x\to 0}\frac{\sin(2x)\sin(5x)}{x^2} = \lim_{x\to 0}\left(\frac{\sin(2x)}{x}\right)\left(\frac{\sin(5x)}{x}\right) =$$

$$= \lim_{x\to 0}\left(\frac{2\sin(2x)}{2x}\right)\left(\frac{5\sin(5x)}{5x}\right) =$$

$$= 2\left(\lim_{x\to 0}\frac{\sin(2x)}{2x}\right)5\left(\lim_{x\to 0}\frac{\sin(5x)}{5x}\right) = 2\cdot 5 = 10$$

8. Let $a > 0$ be a positive real number. Define $f(x) = \begin{cases} x^2 & if \;\; x < a \\ 3x & if \;\; x \geq a \end{cases}$.

What is the value of a if f is continuous on the entire real number line? [10 pts]

If f is continuous on the entire real number line then, in particular, f is continuous at $x=a$.

Thus $\lim_{x\to a} f(x) = f(a)$ or

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

Thus $\lim_{x\to a^-} x^2 = \lim_{x\to a^+} 3x = 3a$. We thus obtain the equation $a^2 = 3a$ which reduces to $\boxed{a=3}$

9. A particle moves along the x-axis such that its position at time t is given by $x(t) = -2te^t$.

a) What is the particle's velocity at time t = 2? [8 pts]

$$x'(t) = -2e^t - 2te^t = -2e^t(1+t)$$

$$x'(2) = -2e^2(3) = -6e^2.$$

b) Is the particle moving right or left? [2 pts]

Since the velocity is negative, the particle is moving to the left

10. Let $f(x) = \dfrac{(x+1)(x^2+4)(x-7)^3}{(x+1)^2(x-7)^2}$.

a) Determine the values of x for which f is continuous. Write your answer in interval notation. [5 pts]

$$(-\infty,-1) \cup (-1,7) \cup (7,\infty)$$

b) For each x where the function is discontinuous, determine if the discontinuity is removable or not. [5 pts]

$$f(x) = \frac{(x^2+4)(x-7)}{x+1} \; ; \; x \neq -1, \; x \neq 7$$

Notice that $\lim_{x \to 7} f(x) = 0$ so the discontinuity is removable at $x = 7$.

Since $\lim_{x \to -1} f(x)$ is undefined, disc. not removable at $x=-1$.

Extra-Credit

11. Prove by means of a delta-epsilon argument that $\lim_{x\to -2}(x^2 - x) = 6$ [10 pts]

$|x^2 - x - 6| = |x+2||x-3|$

let $\delta_0 = 1$

if $0 < |x+2| < \delta_0 = 1$

then $-1 < x + 2 < 1$

$-3 < x < -1$

$-3-3 < x-3 < -1-3 = -4$

$-6 < x-3 < -4$

$6 > |x-3| > 4$

Then if $\delta = \min\{\frac{\epsilon}{6}, 1\}$ we have

$|x^2 - x - 6| = |x+2||x-3| < 6|x+2| < \frac{\epsilon}{6} \cdot 6 = \epsilon.$